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ACCRETION DISCS IN ASTROPHYSICS

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1. INTRODUCTION

If we put a particle in a circular orbit around a central gravitating body, it will stay in that orbit. If we then extract energy and angular momentum from the particle we may allow it to spiral slowly inwards. The amount of energy that can be extracted by such a process is equal to the binding energy of the innermost accessible orbit. For orbits around sufficiently compact objects a reasonable fraction of the particle's rest mass energy can be extracted. For example, of order 10 percent of the rest mass can be obtained from orbits around a neutron star and up to around 40 percent for orbits around a black hole. Thus, the accretion process can be an efficient converter of rest mass to radiation. The problem is to set up the process that can extract the energy and angular momentum.

If we consider a blob of gas in a circular orbit then we have more flexibility. In particular, if we can find a method of redistributing angular momentum among the gas particles in order to let some of them fall into the potential well, then we are in a position to extract the potential energy so released. The accretion disc provides just such a method. The efficiency with which energy is released and the ubiquity of angular momentum explains why accretion discs are popular in models for some of the most luminous objects—X-ray stars and quasars. However, accretion disc theory predates the discovery of both these, and it is to these initial developments that we now turn our attention.

2. INITIAL DEVELOPMENTS

We consider the motion of a rotating mass of gas that is placed in a given potential well (assumed cylindrically symmetric with the axis of symmetry parallel to the angular momentum vector of the gas). We assume that the

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gas can radiate efficiently and that the time scale for any viscous processes (processes that redistribute angular momentum among gas elements) to occur is much longer than both the radiative and the dynamical (orbital) time scales. In these circumstances each element of gas rids itself of as much energy (kinetic plus internal) as it can by colliding with other gas elements, shock heating and radiatively cooling, while retaining its angular momentum. Since the orbit of least energy for a given angular momentum is a circular one, the gas settles down to moving on circular orbits in the form of a thin disc around the central potential.

Adopting cylindrical polar coordinates (R, ϕ, z) the circular velocity $v_\phi \equiv R\Omega(R)$ is given in terms of the potential $\Phi(R)$ on the $z = 0$ plane by

$$v_\phi^2/R = -d\Phi/dR. \tag{2.1}$$

That is, the central gravitational force is balanced by centrifugal force. In general, the fluid rotates differentially, which is to say that the rate of shearing, A, defined by

$$A = R\, d\Omega/dR \tag{2.2}$$

is nonzero.

If there is any viscosity present in the gas, or if there is any other process present that acts in a similar dissipative manner to damp out shearing motions, the energy of the shearing motion is dissipated in the fluid as heat, and thence radiated away. Thus, viscosity causes the gas to lose energy. Since the only energy source is the gravitational potential, this means that the gas sinks deeper in the potential well. Thus, viscosity converts gravitational potential energy into radiation in an efficient manner.

To proceed in a more quantitative manner we must first derive the equations of motion of the gas. Suppose that the disc has surface density $\Sigma(R, t)$ and radial velocity $v_R(R, t)$. We consider the motion of an annulus of gas with inner radius R and with radial extent ΔR. The mass of the annulus is $2\pi R \cdot \Delta R \cdot \Sigma$ and its angular momentum is $2\pi R \cdot \Delta R \cdot \Sigma \cdot R^2\Omega$. The rate of change of the mass of the annulus is equal to the net flow of matter into it from neighboring annuli; that is

$$\frac{\partial}{\partial t}(2\pi R \cdot \Delta R \cdot \Sigma) = v_R(R, t) \cdot 2\pi R \cdot \Sigma(R, t) - v_R(R + \Delta R, t) \cdot 2\pi(R + \Delta R) \cdot \Sigma(R + \Delta R, t). \tag{2.3}$$

Taking the limit for small ΔR, this becomes

$$R\, \partial\Sigma/\partial t + \partial/\partial R\,(R\Sigma v_R) = 0. \tag{2.4}$$

Similarly conservation of angular momentum for the annulus yields the equation

$$R\, \partial/\partial t(\Sigma R^2\Omega) + \partial/\partial R\,(R\Sigma v_R \cdot R^2\Omega) = \mathscr{G}, \tag{2.5}$$

where $\mathscr{G}$ is the net effect of the viscous torques due to neighboring annuli. If the torque of an outer annulus acting on a neighboring inner one at radius R is $G(R, t)$, then

$$\mathscr{G} = \frac{1}{2\pi}\frac{\partial G}{\partial R}. \tag{2.6}$$

The viscous force per unit length around the circumference is $\nu\Sigma A$ where ν is the kinematic viscosity, and hence

$$G(R, t) = 2\pi R \cdot \nu\Sigma A \cdot R. \tag{2.7}$$

Using (2.6) and (2.7), the angular momentum equation (2.5) becomes

$$\partial/\partial t(\Sigma R^2\Omega) + R^{-1}\partial/\partial R\,(\Sigma R^3\Omega v_R) = R^{-1}\partial/\partial R\,(\nu\Sigma R^3\Omega'), \tag{2.8}$$

where here and below prime (′) implies radial derivative. Equations (2.4) and (2.8) may be used to eliminate v_R, yielding the single equation

$$\partial\Sigma/\partial t = R^{-1}\,\partial/\partial R\,\{[(R^2\Omega)']^{-1}\,\partial/\partial R[\nu\Sigma R^3(-\Omega')]\}. \tag{2.9}$$

If the potential is that due to a central point mass M, we have $\Omega = (GM/R^3)^{1/2}$ and (2.9) becomes

$$\partial\Sigma/\partial t = 3R^{-1}\,\partial/\partial R\,\{R^{1/2}\,\partial/\partial R\,[\nu\Sigma R^{1/2}]\}. \tag{2.10}$$

In general since ν may be expected to be a function of local conditions in the disc, that is, a function of Σ, R, and t, Equation (2.9) is a nonlinear diffusion equation for Σ. If, however, ν is a given function of radius the equation is linear in Σ. Moreover, if ν varies as a power of radius the Equation (2.10) can be solved analytically. For example, if ν is constant, Equation (2.10) may be written as an equation for $R^{1/2}\Sigma$ which is separable in terms of the variables $(t, R^{1/2})$. The general solution for a disc extending from $R = 0$ to $R = \infty$ and with zero torque at the origin is

$$\Sigma(R, t) = (12)^{1/4}R^{-3/4}\nu^{-3/4}\int_0^\infty f(\lambda)e^{-\lambda^2 t}J_{1/4}(R\lambda/\sqrt{3\nu})(R\lambda/\sqrt{3\nu})^{1/4}\cdot d\lambda, \tag{2.11}$$

where $f(\lambda)$ is an arbitrary function to be determined from the initial conditions and $J_{1/4}$ is the ordinary Bessel function of order $\frac{1}{4}$. As a particular example we may obtain the Green function, which is the solution corresponding to the initial matter distribution

$$\Sigma(R, t = 0) = m\delta(R - R_0)/2\pi R_0, \tag{2.12}$$

of a ring of mass m at an initial radius R_0. In terms of dimensionless radius $x = R/R_0$ and time $\tau = 12\nu t R_0^{-2}$, the solution is

$$\Sigma(x, \tau) = (m/\pi R_0^2) \cdot \tau^{-1} x^{-1/4} \exp\{-(1 + x^2)/\tau\} \cdot I_{1/4}(2x/\tau) \tag{2.13}$$

where $I_{1/4}$ is the modified Bessel function. The surface density is plotted as a function of time in Figure 1.

It can be seen that the action of viscosity on the initial ring is to spread it out. Most of the mass moves inwards losing energy and angular momentum as it does so, but a tail of matter moves out to larger radii in order to take up the angular momentum. In fact, for the case under consideration the specific angular momentum h for matter in a circular orbit at radius R is given by $h = R^2\Omega \propto R^{1/2}$ and so tends to infinity at large radius. Thus, eventually all of the matter initially in the ring ends up at the origin and all of the angular momentum is carried to infinite radius by none of the mass!

The influence of viscosity on a rotating mass of gas came under scrutiny at an early stage in the consideration of the formation and evolution of the solar nebula with particular regard to Laplace's nebular hypothesis. The general properties of the solution given above—that the dissipative processes act to spread the disc out, allowing the inner parts to move in

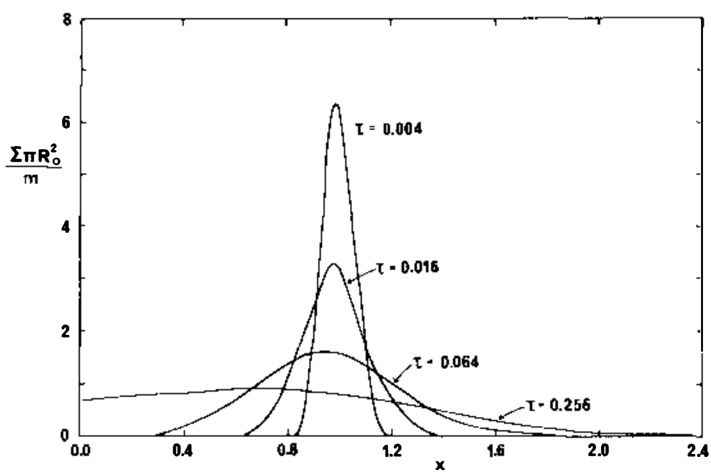

Figure 1 The viscous evolution of a ring of matter of mass m. The surface density Σ is shown as a function of dimensionless radius $x = R/R_0$, where R_0 is the initial radius of the ring, and of dimensionless time $\tau = 12\nu t/R_0^2$ where ν is the viscosity.

and necessitating, through conservation of angular momentum, the outer parts to move out—was well understood by the 1920s. For example, the following is a description by Jeffreys (1924) of the evolution of the early solar nebula:

> The fast-moving interior will tend to drag forward the slower-moving exterior, and thus will increase its energy and make it recede from the sun. Thus the outer parts will be slowly expelled from the system. The inner parts, on the other hand, will have their motion delayed, and will therefore gradually fall into the sun.

In the 1940s, Peek (1942) and von Weizsäcker (1943) came to the conclusion that the effect of turbulent viscosity, acting on the early solar nebula, would be to separate it into two parts—a central core containing most of the mass, and the rest in the form of an infinite disc containing most of the angular momentum. The equations of motion given above were derived by von Weizsäcker (1948) in a paper entitled "The rotation of cosmic gas masses." He also put forward the argument that turbulent viscosity must be the dominant dissipation process and hypothesized a mixing length prescription in which the mixing length varies as a given power of the radius. Such a prescription enables a solution of the form derived above. The general solution was first given in a paper by Lüst (1952) which was dedicated to Heisenberg in honor of his 50th birthday.

3. STEADY DISCS

In the late 1960s interest switched from the secular dynamics of rotating gas masses to the radiation they might emit. The change of emphasis was instigated by the discoveries of, and the appearance of theories to explain, X-ray stars (Prendergast & Burbidge 1968) and quasars (Lynden-Bell 1969). The problem was simplified in that only steady accretion discs were considered, and, although a more sophisticated treatment of radiative processes is now possible, the description of the nature of the viscosity has not progressed to a comparable extent.

In this section I provide a rough sketch of the properties of steady accretion discs. I refer the reader who is interested in a more detailed approach to the seminal paper by Shakura & Sunyaev (1973).

3.1 *Overall Properties*

The general properties of the disc are as described above. The flow takes place in a circular fashion around a central point mass, M. Since the gas can cool readily the circular flow velocity is highly supersonic. In a steady flow the radial momentum equation becomes

$$v_R \frac{\partial v_R}{\partial R} - \frac{v_\phi^2}{R} + \frac{1}{\rho}\frac{\partial p}{\partial R} + \frac{GM}{R^2} = 0, \tag{3.1}$$

where p and ρ are the pressure and density, respectively. The inflow velocity, v_R, is on dimensional grounds of order ν/R and is likely to be subsonic (see below). Thus, if we denote the Mach number of the flow by

$$\mathcal{M} = v_\phi/c_s, \tag{3.2}$$

where $c_s \sim (p/\rho)^{1/2}$ is the sound speed, Equation (3.1) becomes (cf. Equation 2.1)

$$v_\phi = (GM/R)^{1/2}[1 + O(\mathcal{M}^{-2})]. \tag{3.3}$$

Perpendicular to the disc there is essentially no flow and we may apply the equations of hydrostatic equilibrium (von Weizsäcker 1948)

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}\left[\frac{GM}{(R^2+z^2)^{1/2}}\right], \tag{3.4}$$

which for small disc thickness becomes approximately

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{GMz}{R^3}. \tag{3.5}$$

Thus, to order of magnitude the disc thickness H is given by

$$H \sim \mathcal{M}^{-1}R. \tag{3.6}$$

In a steady disc we envisage a constant inward mass flux $\dot{M}$ and noting that $v_R < 0$ obtain from Equation (2.4)

$$\dot{M} = 2\pi R \cdot \Sigma \cdot (-v_R). \tag{3.7}$$

Also in the steady case, the angular momentum equation (2.10) may be integrated to give

$$\nu\Sigma(-\Omega') = \Sigma(-v_R)\Omega - C/(2\pi R^3) \tag{3.8}$$

where C is the constant of integration. At a point in the flow where the shear $A \equiv R\Omega' = 0$, we see that $C = \dot{M}R^2\Omega$. In other words, we may interpret C as the steady influx of angular momentum through the disc. It is usually determined at the inner boundary.

For example, in an accretion disc around a star of radius R_*, the circular velocity remains Keplerian (that is $\Omega^2 = GM/R^3$) all the way down to a narrow boundary layer with width $L \ll R_*$ (Pringle 1977a). Within the boundary layer Ω varies rapidly from a value close to $(GM/R_*^3)^{1/2}$ just outside to the stellar rotation velocity Ω_* at the stellar surface. Thus, the value of Ω where $\Omega' = 0$ is close to the Keplerian value at radius R_*. Neglecting terms of order (L/R) we may write $C = \dot{M}(GMR_*)^{1/2}$, and for a disc around a star we may rewrite Equation (3.8) in the form

$$\nu\Sigma = \frac{\dot{M}}{3\pi}[1 - (R_*/R)^{1/2}]. \tag{3.9}$$

Note that for $R \gg R_*$ we have $v_R = -3\nu/2R$, confirming the order-of-magnitude estimate made above.

From standard fluid dynamics, the kinematic viscosity ν generates dissipation in the disc at a rate $D(R)$ per unit area per unit time, where

$$\begin{aligned} D(R) &= \tfrac{1}{2}\nu\Sigma(R\Omega')^2 \\ &= \frac{3GM\dot{M}}{4\pi R^3}[1 - (R_*/R)^{1/2}]. \end{aligned} \tag{3.10}$$

To obtain the final expression we have used Equation (3.9). This simple result underlines the attractive simplicity of steady disc theory. The major uncertainty of the theory—the viscosity—has vanished, although this is at the expense of the assumption that the viscosity can indeed adjust itself to provide the steady mass flux $\dot{M}$. The result is not a surprising one since the local dissipation rate for a given steady accretion rate $\dot{M}$ is just determined by the potential drop (that is, by M and R). Of course, the other local disc properties—for example the surface density, optical thickness, and so on—do depend on the size of the viscosity (see Equation 3.9). Using (3.10) we find that the total disc luminosity is

$$\begin{aligned} L_{\text{disc}} &= \int_{R_*}^{\infty} D(R) \cdot 2\pi R\, dR \\ &= \tfrac{1}{2}GM\dot{M}/R_*. \end{aligned} \tag{3.11}$$

This is only one half of the total available accretion energy, since the total potential drop from infinity to radius R_* is GM/R_*. The discrepancy arises because the matter just outside the boundary layer still retains one half of the potential energy it has lost as kinetic energy. Thus, the rest of the accretion luminosity is emitted in the boundary layer. This implies that a study of the details of emission from the inner edge of an accretion disc can be just as important as studying emission from the disc itself.

3.2 *Viscosity*

Before discussing the details of steady discs we should turn our attention briefly to the nature of the viscosity, on which the details implicitly depend. Section 4 discusses at greater length the various viscous mechanisms that have been suggested. The most fruitful approach to the problem has been the condensation of our uncertainties by Shakura & Sunyaev (1973) into a dimensionless parameter α.

Suppose that the dominant process for redistributing angular momentum is a turbulent viscosity of some kind. (It should be noted that despite the high Reynolds number there is as yet no proof that accretion discs are unstable to turbulent motions.) The effective kinematic viscosity of a turbulent process is usually written in the form

$$\nu \sim \ell \cdot v \tag{3.12}$$

where ℓ is the size and v the turnover velocity of the largest eddies. For turbulence in an accretion disc we may surmise that the scale of the eddies is less than the disc thickness H (certainly true) and that the turbulence is subsonic (probably true). Thus, we may write

$$\nu = \alpha c_s H \tag{3.13}$$

where c_s is the sound speed in the disc and $\alpha \leq 1$. Shakura & Sunyaev also show that magnetic stresses due to a tangled field in the disc give rise to a similar formula with $\alpha \sim v_A^2/c_s^2$ where v_A is the Alfvén speed in the disc.

With this parametrization it is worth noting that the radial velocity $v_R \sim \alpha \mathcal{M}^{-1} c_s$ is, as anticipated, highly subsonic.

3.3 *Local Disc Structure*

Once a formula for the viscosity is given, we can calculate the detailed local structure of the disc. If the disc is isothermal in the z-direction at given radius R, then Equation (3.5) integrates to give

$$\rho(z, R) = \rho_c(R) \exp(-z^2/2H^2) \tag{3.14}$$

where $H^2 = c_s^2 R/GM$. Thus, the density falls off rapidly with scale height above the disc. It is therefore appropriate to treat the local structure of the disc as if it is governed by the properties of the disc on the $z = 0$ plane. To show what is involved without entering into confusing detail, I shall proceed in a simple manner and ignore quantities of order unity. All quantities are assumed to be evaluated at $z = 0$ except that it is necessary to distinguish between central and surface temperatures (T_c and T_s) at each radius. Thus we define disc density by

$$\rho = \Sigma/H \tag{3.15}$$

where scale height H is now given simply by

$$H = R(c_s/v_\phi). \tag{3.16}$$

Since the disc is thin ($H \ll R$), ∇T is in the z-direction to first order in H/R so that we need only consider radiative transfer perpendicular to the disc. Thus the structure of the disc in the z-direction at a given radius is similar to that of a one-dimensional star and is determined by a balance between heat input per unit area,

$$Q^{+} = \tfrac{1}{2}D(R) \tag{3.17}$$

(the factor of one half comes from considering one side of the disc only), and heat loss per unit area, which is given by the radiative transfer equation roughly as

$$Q^{-} = acT_{c}^{4}/\tau. \tag{3.18}$$

Here a is Stefan's constant and τ, the optical depth, is given by

$$\tau = \rho H \kappa(\rho, T_{c}) \tag{3.19}$$

where κ is the opacity.

Equating Q^{+} and Q^{-} and using the viscosity prescription given by Equation (3.13) together with a constitutive relation of the form

$$c_{s}^{2} = p/\rho \tag{3.20}$$

where

$$p = \frac{1}{3} a T_{c}^{4} + \frac{\mathcal{R} \rho T_{c}}{\mu} \tag{3.21}$$

completes the set of equations and enables us to solve for ρ, T_c, c_s, H, and Σ as functions of $\dot{M}$, M, R, and the unknown viscosity parameter, α.

3.4 *The Emitted Spectrum*

To determine the emitted spectrum of the disc we must determine the spectrum emitted locally at each part on the disc surface and then integrate over the whole disc surface. For a proper local determination of the emitted spectrum, we should solve the radiative transfer equation properly at each radius. If the disc is optically thick, in the sense that each element of the disc radiates as a black body with temperature $T_s(R)$, the problem simplifies enormously. We then have

$$Q^{-} = \sigma T_{s}^{4} \tag{3.22}$$

where σ is the Stefan-Boltzman constant, and the only reason for calculating the local disc structure is to verify the blackbody assumption. Putting $Q^{+} = Q^{-}$ we find

$$T_{s} = \left\{ \frac{3GM\dot{M}}{8\pi R^{3}\sigma} \left[1 - (R_{*}/R)^{1/2}\right] \right\}^{1/4}. \tag{3.23}$$

We note that for $R \gg R_*$, $T_s = T_*(R/R_*)^{-3/4}$ where
$T_* = (3GM\dot{M}/8\pi R^3 \sigma)^{1/4}$.

At each radius the spectrum emitted by an elemental area of the disc is the blackbody spectrum

$$B_\nu(T_s) \propto \nu^3[\exp(h\nu/kT_s) - 1]^{-1}. \tag{3.24}$$

Thus, the spectrum given by the disc as a whole is then

$$S_\nu \propto \int_{R_*}^{R_{out}} B_\nu[T_s(R)] \cdot 2\pi R\, dR \tag{3.25}$$

where R_{out} is the outer disc radius. This spectrum is shown in Figure 2.

The functional form is relatively easy to understand. For frequencies $\nu \gg kT_*/h$ the spectrum drops exponentially as all we see is the exponential tail of the spectrum emitted by the hottest disc elements. For $\nu \ll kT_*/h$, the radiation comes predominantly from radii $R \gg R_*$ so that we may take $T_s \propto R^{-3/4}$. We can then rewrite the integral in the form

$$S_\nu \propto \nu^{1/3} \int_0^{x_{out}} \frac{x^{5/3}\, dx}{e^x - 1}.$$

Here $x = h\nu/kT_s(R)$, $x_{out} = h\nu/kT_{out}$, and $T_{out} = T_s(R_{out})$. Thus, for frequencies such that $kT_{out}/h \ll \nu \ll kT_*/h$ we may take $x_{out} \gg 1$ and find $S_\nu \propto \nu^{1/3}$. This is sometimes regarded as a characteristic accretion disc spectrum and was first noted by Lynden-Bell (1969). For $\nu \ll kT_{out}/h$,

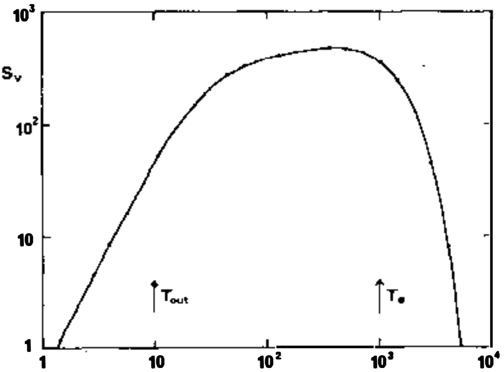

Figure 2 The integrated spectrum of a steady accretion disc that radiates a local blackbody spectrum at each point. The units are arbitrary, but the frequencies corresponding to T_{out}, the temperature of the outermost disc radius, and to T_*, the characteristic temperature of the inner disc, are marked.

the Rayleigh-Jeans tail of the coolest disc elements dominates and we find $S_\nu \propto \nu^2$. We note that the hottest temperature in the disc T_{max} occurs at radius $R = 49/36\ R_*$ and is equal to $T_{max} = 0.488T_*$.

3.5 *Breakdown of Thin Disc Approximation*

Though we have assumed so far that the disc is thin, that is $H \ll R$, there is no reason why this should always be so. For example, if the radiative process is inefficient relative to the local rate of dissipation of energy then the disc may be thick. If the disc is thick, so that $H \sim R$ and $\mathcal{M} \sim 1$ then all the thin disc approximations break down simultaneously. In particular, we may no longer neglect the radial pressure gradient, nor may we assume that $\partial T/\partial z \gg \partial T/\partial R$, and neglect heat flow in the radial direction. Thus, the disc is no longer Keplerian, and the splitting of the solution of disc structure into a radial part and a part perpendicular to the disc plane is no longer valid. The disc behaves as if it is a rapidly rotating centrally condensed star and should, strictly, be treated accordingly.

4. VISCOSITY

The main failing of accretion disc theory is that it has no predictive power except in certain limiting circumstances. The main reason for the lack of predictive power is the uncertainty as to the nature and magnitude of the viscosity. As we have seen, the viscosity governs the local structure of the disc as well as the time scale on which it evolves. Given a viscosity and a radiative process we are set to build an accretion disc model. The only reasonable certainty is that ordinary molecular viscosity is too small to be of consequence, although there is no reason why models using such a viscosity cannot be built. In this section we discuss the various forms of viscosity that have been put forward.

The problem is that we are dealing with a strongly shearing medium that is highly supersonic, is radiative, and has a large Reynold's number. The shear and the high Reynold's number have tempted authors to conclude that the flow must be unstable to turbulent motions. This led to the early discussions of mixing length theory (von Weizsäcker 1948) and was part of the motivation for the α-prescription (Shakura & Sunyaev 1973). Note that in α-discs the effective Reynolds number is

$$\mathrm{Re} \sim \alpha^{-1}(R/H)^2 \gg 1.$$

Lynden-Bell & Pringle (1974) have argued that $\mathrm{Re} \sim 10^3$, based on intuition from laboratory turbulence (subsonic and nonradiative), but there is no justification for such an assertion. Despite various attempts (see, for example, Stewart 1975) there is as yet no evidence that the flow is turbulently unstable. The main problem is that the Rayleigh criterion for stability,

$$\frac{d}{dR}(R^2\Omega) > 0,$$

is amply satisfied (Safronov 1969).

One of the more plausible viscous processes likely to be operating is magnetic viscosity (Lynden-Bell 1969, Shakura & Sunyaev 1973)—that is the transfer of angular momentum by magnetic stresses. Consider the effect of the disc flow on a small tangled magnetic field embedded in the disc. The radial component of the seed field is amplified by the shear into an azimuthal component with a growth time scale of Ω^{-1}. Once the magnetic pressure is comparable to the central disc pressure, the field lines are unstable to bowing out of the disc (Parker 1970, Shu 1974). Reconnection can then occur in the less dense regions out of the plane of the disc, leading perhaps to regeneration of a poloidal component of the field and thus completing the dynamo cycle. As mentioned above, Shakura & Sunyaev (1973) have pointed out that a magnetic viscosity can be accommodated within the α-prescription. Further work has been done by Eardley & Lightman (1975), Ichimaru (1976), and by Pudritz (1980). One problem with these ideas has been emphasized by Pustilnik & Shvartsman (1974) who note that the Parker instability tends to encourage disc material to form into discrete blobs, each threaded by magnetic field. It remains to be shown whether the shear can smooth out such blobs on a short enough time scale that the disc may be treated as a smooth, homogeneous entity or whether real discs are lumpy (Pringle 1977c).

A more radical approach to the idea of using magnetic fields to transfer away angular momentum has been made by Blandford (1976) and by Lovelace (1976); see also Blandford & Znajek (1977). They show that if it can be arranged that the disc has embedded in it an ordered magnetic field with a sufficiently large component perpendicular to it, then the disc can act as a kind of two-dimensional pulsar. The energy and angular momentum of the disc material can be carried away directly in the form of a magnetized relativistic wind.

A possible source of turbulent motions that might lead to angular momentum transfer occurs if the disc is convectively unstable. The criterion for convective instability differs slightly from the usual Schwarzschild criterion and is discussed by Livio & Shaviv (1977) and by Tayler (1980).

In accretion discs relevant to optical or X-ray observations, convection is liable to occur if the surface temperature is sufficiently low ($T_s \lesssim 10^4$ K) that there is a hydrogen or helium ionization zone within the disc. Otherwise, whether a disc is convective or not depends on the details of where in the disc energy is deposited (Stewart 1976, Bisnovatyi-Kogan & Blinnikov 1977, Shakura et al. 1978, Tayler 1980). For discs occurring in close binary systems or in quasars, convection is unlikely to be a major source of viscosity. Models using convective motions as a viscosity are given by Vila (1978) and by Liang (1977b). In cooler accretion discs, however, for example in the early solar nebula where other opacities (e.g. dust) are dominant, convection may be able to provide a major source of viscosity (Cameron 1978, Lin & Papaloizou 1980).

One known source of viscosity is electron viscosity in material supported by electron degeneracy pressure. Disc models using this have been constructed by Paczynski & Jaroszynski (1978).

The criterion that the self-gravity of an accretion disc is negligible is given simply by the demand that the dominant gravity force containing the disc in the z-direction is that due to the central object, mass M. The criterion can be written in terms of the disc surface density

$$\Sigma \ll \frac{MH}{R^3}$$

or, writing the disc mass M_d within radius R as roughly $M_d \sim \Sigma R^2$, this becomes

$$M_d/M \ll H/R.$$

If self-gravity becomes important, the disc may become unstable with the fastest growing perturbations being of size $\sim H$. Paczynski (1978a) has argued that in certain circumstances the disc may hover on the brink of gravitational instability. He envisages a disc in which gravitationally unstable clumps are sufficiently opaque that collapse is possible only on a thermal time scale that is much longer than the disc's dynamical time scale of Ω^{-1} (see Section 6). He postulates that this state of affairs gives rise to turbulence which if violent enough provides a feedback mechanism to heat the disc (increase H) and shut off the instability. In this event the prescription for the viscosity is replaced by the requirement that self-gravity be marginally important, that is $\Sigma \simeq MH/R^3$ (Paczynski 1978a, Kozlowski et al. 1979). Such a possibility has also been envisaged for galactic discs (Goldreich & Lynden-Bell 1965; see also the discussion by Toomre 1977).

A simpler consideration of viscosity is possible in a disc that consists of nearly collisionless particles. Suppose the particles have velocity dispersion v and collision frequency τ_c^{-1}. If the collision frequency is much more than the orbital frequency Ω, then the standard formula for the kinematic viscosity holds; that is, $\nu \sim v\lambda_c$ where $\lambda_c = v\tau_c$ is the collision mean free path. If, on the other hand, the collision frequency is small, the particles orbit many times before colliding and the effective mean free path is reduced from λ_c to the mean radial excursion $\lambda_R \sim v\Omega^{-1}$. Further, the particles describe the radial excursion $\sim\tau_c\Omega$ times before colliding. Thus the viscosity is reduced to $\nu \sim v\lambda_R(\tau_c\Omega)^{-1} \sim v^2\tau_c(\tau_c\Omega)^{-2}$. Note that the disc thickness $H \sim v\Omega^{-1} \sim \lambda_R$ so that the viscosity is reduced as soon as the mean free path to collisions significantly exceeds the disc thickness. Such nearly collisionless viscosities have been considered by Paczynski (1978b) in a hot disc model involving ion viscosity and bremsstrahlung cooling. They are also relevant to discs that consist of small bodies that collide inelastically, for example in models of Saturn's rings (Brahic 1977, Brahic & Henon 1977) and of the early solar system (Franklin et al. 1980).

In the long run it is probable that an observational approach to discovering the properties of disc viscosity will be more fruitful than the purely theoretical one. As usual, however, it is not easy to interpret the observations in an unambiguous manner. The only systems in which accretion discs are known to exist and in which their time-dependent behavior can be monitored are the cataclysmic variables. The outbursts of dwarf novae, and possibly of some recurrent novae, are accretion events and the time scale for variability during outburst enables estimates of the viscosity to be made. In his model for the outburst of T CrB, Webbink (1976) finds that the observations imply an effective Reynolds number Re in the disc of the order of Re $\sim 10^3$. For dwarf nova outbursts, values of Re in the range 10^2–10^3, or alternatively values of α in the range 0.1–1 provide reasonable fits to the data (Lynden-Bell & Pringle 1974, Bath & Pringle 1981). Whether the viscosity always remains at that level or falls to a lesser value during quiescence is still a matter for debate (Bath 1978, Mallama & Trimble 1978).

5. RADIATION MECHANISMS

Although the magnitude of the viscosity obviously affects the time scale on which a disc can vary, it is conceivable that the spectra emitted by steady accretion discs, in which the radiation per unit area does not depend on viscosity, does not depend too strongly on the nature of the dissipative process. In the steady discs considered by Shakura & Sunyaev (1973) in which dissipation occurs within the disc at several optical depths from the emitting surface, the outgoing spectrum is indeed relatively independent of the viscosity parameter α. If, however, the disc is optically thin, or if dissipation takes place in optically thin parts of the disc, whether in thin patches between blobs or in an extended hot corona out of the plane of the disc, the nature of the dissipation and the details of the emission process and of the geometry can be important. A variety of proposals have been advanced in which, sadly, the uncertainty of the theory does not detract from the possibility of relevance of the models. We turn our attention to these below, but consider first the simpler case of a steady, optically thick, homogeneous disc in which dissipation occurs within the body of the disc at the rate prescribed by Equation (3.10).

The spectrum from such a disc in which the surface emissivity is that of a blackbody was considered in Section 3. To test the validity of the blackbody assumption it is necessary to construct a detailed disc, essentially building a one-dimensional stellar atmosphere at each radius. The modification to the spectrum due to the importance of electron scattering as an opacity source is considered by Shakura & Sunyaev (1973) who show that this tends to flatten the spectrum from the blackbody slope of $\nu^{1/3}$. More recently, interest has centered on the modeling of the accretion discs seen in cataclysmic variables. Simple models, assuming local blackbody emissivity, have been carried out by Evans (1974), Bath (1976), Schwarzenberg-Czerny & Rozyczka (1977), and Tylenda (1977). More sophisticated approaches have been adopted by Herter et al. (1979), Kiplinger (1979), and by Mayo et al. (1980) who use model stellar atmospheres to predict the flux at each point. This method also enables predictions to be made of the strengths and profiles of absorption lines in the spectrum (Koen 1976, Mayo et al. 1980). Predictions of absorption line profiles in the X-ray region of the spectrum have been made by Mészáros (1974). There is some initial evidence that the spectra emitted by the discs in dwarf novae are in accord with the predictions (Bath et al. 1980).

If a disc is saucer shaped, that is, if $d(H/R)/dR > 0$, it may be necessary to take into account radiation emitted from the central regions that is reabsorbed by the outer parts. The ratio of reabsorbed radiation to that dissipated locally is given approximately by $(R/R_*) \times d(H/R)/d \log R$ and is usually small for thin discs (Evans 1974, Pacharintanakul & Katz 1980). However, although this means that the continuum flux is not strongly affected, if the central flux is absorbed above the local disc photosphere, then emission lines can be produced. An early study of emission-line formation in accretion discs has been made by Gorbatskii (1965). More recently Williams (1980) has studied emission-line formation in the cool, outer, and optically thin, parts of discs in dwarf novae.

If the viscous mechanism operating in an accretion disc does not dissipate the energy in a region of sufficiently high optical depth, then the emergent radiation is not fully thermalized, and its characteristic temperature is higher than that predicted by the optically thick disc models. A strong motivation for much of the work on nonthermal and optically thin emission from accretion discs is provided by the observations of the X-ray source Cygnus X-1. There is reasonable evidence that the X-ray source is powered by disc accretion onto a black hole, but the observed X-ray spectrum is too hard to be produced by the standard, optically thick disc models. To circumvent this problem it is necessary to hypothesize that the energy is dissipated predominantly in parts of the disc that are optically thin.

The "two temperature" disc model was proposed to this end by Eardley et al. (1975) and Shapirc et al. (1976). In this model the inner disc regions are hot and optically thin and the predominant cooling mechanism is via Compton scattering of an abundant flux of soft photons incident from elsewhere—perhaps from the outer cooler parts of the disc. This radiation mechanism may be of importance in other branches of astronomy and has been the subject of further research [see, for example, Katz (1976) and references therein].

If the predominant disc viscosity is magnetic then it is reasonable to expect that most of the dissipation takes place in the less dense regions out of the plane of the disc where the Alfvén speed is higher and field reconnection can occur more rapidly. This could lead to the formation of a strongly radiating hot corona above and below the disc. A variety of cooling mechanisms is available to potential disc modelers, including bremsstrahlung, Compton scattering of cool disc photons, and cyclotron and/or synchrotron radiation from the hot electrons in the reconnecting magnetic field. Coronal disc models have been applied to X-ray bursters (Liang 1977a), Cygnus X-1 (Thorne & Price 1975, Liang & Price 1977, Blinnikov & Bisnovatyi-Kogan 1978, Galeev et al. 1979), and to the production of γ-rays in Cygnus X-3 (Mészáros et al. 1977).

Various other radiation processes have also been considered. Models of discs around massive black holes emitting predominantly by the synchrotron process have been constructed by Lynden-Bell (1969) and by Pringle et al. (1973). The latter authors also considered optically thin discs emitting by bremsstrahlung. Accretion discs as plasma turbulent reactors have been considered by Norman & ter Haar (1973).

Rees (1975) and Lightman & Shapiro (1975) have estimated the polarization to be expected for radiation from X-ray accretion discs. For example, if in a thin disc the predominant opacity is electron scattering, we might expect the light to be slightly polarized with the electric vector lying in the plane of the disc. On the other hand, if the emission comes from a quasi-spherical cloud of hot gas, little polarization is expected, if any. Thus, polarization may provide some information about the geometry of the emitting regions.

If the accretion rate in a steady disc is such that the total disc luminosity $GM\dot{M}/R_*$ would exceed the limiting Eddington luminosity L_E, then the disc thickness becomes of order of the radius at a point at which $R/R_* \sim GM\dot{M}/(R_*L_E)$. It is speculated that from within that radius the excess accreting matter is blown off in the form of a wind (Shakura & Sunyaev 1973, Wickramasinghe 1974, Pringle & Webbink 1975). Even when the disc is radiating at close to, but below, the Eddington limit, the thin disc approximation breaks down in the inner regions and, for example, radial radiation pressure must be taken into account (Maraschi et al. 1976, Maraschi & Treves 1977, see also Section 9).

At low luminosities it is still possible that accretion discs emit winds in much the same way that stars are seen to do. The mass loss rate for a disc, however, is literally infinitely more unpredictable than that for a star. A spherical star has just an unpredictable mass loss rate. A disc has an unpredictable mass loss rate which is a function of radius in the disc. Piran (1977) has estimated mass loss rates using simple evaporative considerations. Bisnovatyi-Kogan & Blinnikov (1977) and Icke (1977) have estimated the structure of a disc wind flow by calculating the orbits of particles driven by radiation from the disc. Hydrodynamic models of the flow have been constructed by Bardeen & Berger (1978), Meier (1979), and Icke (1980).

6. TIME DEPENDENCE

In Section 2 we derived the equation for the evolution of the surface density Σ of an accretion disc. For a disc around a central point source the equation is (2.10)

$$\partial\Sigma/\partial t = 3R^{-1}\,\partial/\partial R[R^{1/2}\partial/\partial R(\nu\Sigma R^{1/2})]. \tag{6.1}$$

As we mentioned, if the viscosity ν varies as a given power of radius the equation can be solved analytically (Lüst 1952; see also Lynden-Bell & Pringle 1974). The solutions fall into two classes. The first involves Bessel functions J_k with $k > 0$, and have zero torque at the origin. These represent material draining into the origin, transferring a fixed quantity of angular momentum out to infinity. The second is similar but with $k < 0$ and has nonzero torque at the origin. The total amount of mass in the disc now stays constant and it is all eventually dispersed to infinity by being given angular momentum at the origin. Such a solution might represent a disc around a magnetized star which is rotating sufficiently rapidly that its angular velocity exceeds the Keplerian value at the magnetosphere.

In general the viscosity is likely to depend on local parameters in the disc. In its simplest form we may write $\nu = \nu(\Sigma, R)$ and Equation (6.1) becomes a nonlinear diffusion equation. Barring instabilities, this does not change the character of the solutions. For example, if the viscosity is of the form $\nu \propto \Sigma^a R^b$, as is indeed the case for an optically thick α-disc ($a = \frac{3}{7}$, $b = \frac{15}{14}$), then analytic solutions of (6.1) can be constructed (Pringle 1974) in a manner similar to that described by Zel'dovich & Raizer (1966) for the solution of nonlinear conduction.

Time-dependent accretion discs with α-parameter viscosity were constructed by Lightman (1974a,b). To do so he showed the importance of considering the relative magnitudes of the various time scales present in the disc. At a particular radius, the basic disc time scale is given by the rotational angular frequency,

$$t_\phi \equiv R/v_\phi = \Omega^{-1}. \tag{6.2}$$

This is the shortest time scale present in the disc. The time scale on which hydrostatic equilibrium is established in the z-direction is given by

$$t_z = H/c_s. \tag{6.3}$$

From Equation (3.6) we see that $t_z \simeq t_\phi$. The time scale on which changes occur in local surface density is the viscous time scale,

$$t_\nu = R^2/\nu, \tag{6.4}$$

and for the α-prescription we have

$$t_\nu \sim \alpha^{-1}(R/H)^2 t \gg t_\phi. \tag{6.5}$$

We conclude, first, that the assumption of circular Keplerian velocities used in obtaining (6.1) is valid and, second, that for time-dependent α-discs it is sufficient to construct a static atmosphere in the z-direction at each radius at each point in the slow viscous evolution.

The models constructed by Lightman were of α-discs around neutron stars and black holes with a view of investigating the properties of binary X-ray stars. In doing so he uncovered some inconsistencies in the α-prescription (see Section 7). More recently Bath & Pringle (1981) have constructed similar models around white dwarf stars to investigate the various models for the outbursts in dwarf novae. More simplistic models of time-dependent disc behavior have been previously applied to dwarf novae (Bath et al. 1974b) and to the recurrent nova T CrB (Webbink 1976).

Various authors have considered precessing accretion discs in models for the objects Her X-1 and SS433 among others. For this purpose it is necessary to consider a disc that is nonplanar. The equations for the evolution of nonplanar discs, and solutions, are given by Petterson (1977a,b, 1978) and Merritt & Petterson (1980). It should be noted that in such models the uncertainties inherent in the viscosity are magnified because it is now necessary to specify both the (R, ϕ) and the (R, z) components of the stress tensor.

The solution of the more general Equation (2.9) for the evolution of a disc in an arbitrary potential well has been discussed by Icke (1979) who considers the evolution of a gaseous disc in the potential well of a galaxy. He notes that the viscous time scale, for a given viscosity, depends on the local rate of shearing (that is on Ω') and that gas can, therefore, accumulate at radii where Ω' is small.

The time-dependent behavior of discs has also been considered with reference to Saturn's rings using inelastic particle collisions as a viscous process (Brahic 1977, Brahic & Henon 1977).

To complete the cycle from the early investigations of von Weizsäcker and Lüst, time-dependent models of the early solar nebula are being investigated in which α-type models of the viscosity are used and in which the local dissipation, radiation loss, and the z-structure are treated in a manner similar to that described above (Cameron 1978, Lin & Papaloizou 1980). For the material and temperatures in the early solar nebula, the opacity is a complicated, sensitive, and not completely known function of temperature and depends largely on the presence of grains and their melting temperatures. Lin & Papaloizou (1980) argue that opacity-induced convection can be a major viscosity in the early solar nebula and that it is even possible that the various discontinuities in the opacity law give rise to aggregation of matter at particular radii and hence to the positions of the major planets.

7. INSTABILITY

We have assumed so far that the discs under discussion are stable in the sense for a steady disc that small perturbations of the equilibrium solution do not grow, or for a slowly evolving disc that the density behaves in a sensible manner and does not, for example, become singular anywhere. Before considering the details it should be noted that the term "instability" can be misleading in this context. For example, if we construct an equilibrium model of a steady disc and the model turns out to be unstable in the above sense, then the first conclusion to draw is not that we have discovered some physical instability but rather that one of our initial assumptions made in order to construct the disc (perhaps the nature of the viscosity or the type of radiation process) is inconsistent with the assumption of steadiness. Thus, the "instabilities" of accretion discs discussed here and in the literature can more properly be regarded as "self-inconsistencies."

Equilibrium models of accretion discs are subject to two types of instability—viscous instabilities discovered by Lightman & Eardley (1974) and thermal instabilities discussed initially by Pringle et al. (1973). A thorough investigation of such instabilities was made by Shakura & Sunyaev (1976). They found that instability develops through a bifurcation of unstable modes—the viscous and the thermal. We consider each in turn.

The thermal time scale $t_{\rm th}$ of an accretion disc is given by the heat content per unit area divided by the local dissipation rate, that is

$$t_{\rm th} \simeq \frac{\Sigma c_s^2}{D(R)}. \tag{7.1}$$

From Equation (3.10) we see that $D(R) \simeq \nu\Sigma\Omega^2$ and hence obtain

$$t_{\rm th} \simeq (H/R)^2 t_\nu \ll t_\nu. \tag{7.2}$$

In fact, for an α-disc, with $\alpha \lesssim 1$, we may write

$$t_{\rm th} \simeq \alpha^{-1} t_\phi \geq t_\phi. \tag{7.3}$$

Thus, the thermal instability is the faster growing of the two. This fact also enables us to consider it more simply.

Because $t_{\rm th} \ll t_\nu$, we may treat the local surface density Σ as fixed and just consider the variation with time of disc thickness H, or equivalently T_c, at a given radius. Moreover, because $t_{\rm th} \gtrsim t_\phi = t_z$, we may assume that the disc remains in hydrostatic equilibrium in the z-direction as the instability develops. In equilibrium, the local disc heating rate Q^+ exactly balances the local disc cooling rate Q^-. If Σ is fixed and hydrostatic equilibrium holds in the z-direction, then at a given radius Q^+ and Q^- can be written as functions of T_c alone. It is then evident that the condition for thermal instability is (Pringle 1977b)

$$\text{Instability} \Leftrightarrow \frac{d\log Q^+}{d\log T_c} > \frac{d\log Q^-}{d\log T_c}, \tag{7.4}$$

because if the criterion holds, then a small increase (decrease) in T_c leads to a heating rate that is more than (less than) the cooling rate and hence to a further increase (decrease) in T_c.

To understand the viscous instability we make use of the fact that during its slow growth we may assume, since $t_\nu \gg t_{\rm th} \gtrsim t_z$, not only hydrostatic but also thermal equilibrium holds in the z-direction. That is we may take $Q^+ = Q^-$. Using these assumptions, the viscosity at a given radius may be written as a function of Σ alone (we made use of this fact in Section 6), and we have $\nu = \nu(\Sigma, R)$. For convenience we write $\mu \equiv \nu\Sigma$. If we assume that the disc is initially in equilibrium and perturb the surface density in the form $\Sigma \to \Sigma + \delta\Sigma$, and hence $\mu \to \mu + \delta\mu$ where $\delta\mu = (\partial\mu/\partial\Sigma)\,\delta\Sigma$, the evolution equation (6.1) can be written

$$\frac{\partial}{\partial t}(\delta\mu) = \frac{\partial\mu}{\partial\Sigma}\cdot\frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(R^{1/2}\delta\mu)\right]. \tag{7.5}$$

This is a linear diffusion equation for $\delta\mu$ and is well behaved if and only if the diffusion coefficient is positive. Thus the criterion for viscous instability is

$$\text{Instability} \Leftrightarrow \frac{\partial}{\partial\Sigma}(\nu\Sigma) < 0 \tag{7.6}$$

where the derivative is taken under the assumptions of vertical hydrostatic equilibrium and of thermal balance. If a disc is viscously unstable, a region that is locally underdense (overdense) evolves faster (slower) than its surroundings and thus becomes even more underdense (overdense).

Further discussion of instabilities in accretion discs can be found in Shibazaki & Hoshi (1976) and Livio & Shaviv (1977). Detailed instability criteria are to be found in Piran (1978). For an alternative approach see Kato (1978).

As we noted above, the physical meaning of such instabilities is unclear. For example, it is easy to show that the central regions of the standard α-discs in which radiation pressure and electron-scattering opacity dominate are unstable to both thermal and viscous instabilities. The α-viscosity law is $\nu = \alpha c_s H$ which can be written $\nu = \alpha c_s^2/\Omega$. Lightman & Eardley noted that changing the viscosity law (which is anyway ad hoc) by letting $c_s^2 = P_g/\rho$ where P_g is the gas pressure, rather than taking $c_s^2 = (P_r + P_g)/\rho$ where P_r is the (dominant) radiation pressure, renders the disc stable.

However, while it may always be possible to invent stable viscosity laws for any given situation, it is also possible, though unproven, that in some circumstances a steady disc does not form.

8. VARIABILITY

The shortest time scale present in the disc at a given radius is the time taken for a particle in the disc to orbit the central body and is given by $t_\phi \simeq \Omega^{-1}$. Thus although a disc can obviously vary on the viscous time scale due to a change in boundary conditions (for example, a variation in accretion rate), if the disc is unsteady we might expect the radiation it gives out to vary on all time scales down to the value of Ω^{-1} at the innermost disc radius.

It is not clear, however, that a strong long-lasting periodic variation in luminosity can be obtained from an accretion disc. (Here I exclude discussion of periodicities occurring in the boundary layer at the inner disc radius; see Papaloizou & Pringle 1978.) The reason is that any azimuthal variation in emissivity that might produce a periodic variation in brightness is rapidly erased by the strong shear present in the disc. Models that do not take the shear into account in disc variability (for example Kato 1978, Van Horn et al. 1980) are, therefore, incomplete.

As an illustration, suppose that for some reason a blob forms in the disc which is significantly more luminous than its surroundings. Then the blob orbits the central object with period $P = 2\pi/\Omega$ and if the geometry is such that our view of it varies round its orbit [for example, if the disc is saucer-shaped or if the blob is orbiting sufficiently close to the center that either relativistic effects are important (Sunyaev 1973) or it suffers eclipse by the central gravitating body (Bath et al. 1974a)] the period can show up as a regular variation in the luminosity. Suppose, however, that the blob when formed has a size ΔR in the radial direction and $R\,\Delta\phi$ in the azimuthal direction. We expect that $H \leq \Delta R \leq R$ and $H \leq R\,\Delta\phi \leq R$. Then two comments are in order. First, if the blob formation process has no preferred azimuth, there is no reason for just one blob to form at a given radius, but rather we expect $N \sim 2\pi/\Delta\phi$ blobs to form. Since we are using geometrical effects to render the blobs visible we may therefore expect the luminosity variation to be reduced by a factor N. Second, even if only one blob is present we may expect the effect of the blob (assuming it has infinite cooling time) to last only until it is sheared out by azimuthal motions. One blob lasts until it is sheared out to a distance $\sim R$ in azimuthal extent—a time of order $\Omega^{-1}R/\Delta R$. Thus the Q of the periodic luminosity variation is at most $Q \sim R/\Delta R$ and since $\Delta R \geq H$ we have $Q \leq R/H$. If, however, there are N blobs, then Q is reduced to $Q \sim R\,\Delta\phi/\Delta R$.

If we form a blob of radial extent ΔR, then we may expect it to give rise to a luminosity variation of at most $\Delta L/L \sim \Delta R/R$. From the above, we see that to maximize Q and to minimize N we would like $\Delta\phi \sim 1$, in which case $Q \sim L/\Delta L$. Thus, for a periodic variation in an accretion disc large amplitude and high Q are incompatible requirements.

Moreover, since the disc is highly supersonic it cannot communicate efficiently in the azimuthal direction. This implies that without the help of some globally acting outside agency—for example an oscillating star at the inner radius or a global magnetic field—it is difficult to conceive of a process that would produce blobs with $R\,\Delta\phi$ much exceeding H. In that case we expect $Q \sim 1$ and $\Delta L/L \sim (H/R)^2$.

9. INNER BOUNDARY

In Section 3 we commented on the relevant inner boundary condition for a disc around a slowly rotating star. We argued that $\Omega' = 0$ at a radius close to $R = R_*$ and at a value of Ω close to the Keplerian value $\Omega_K = (GM/R_*^3)^{1/2}$. For a nonrotating star, one half of the accretion luminosity comes from the boundary layer at the inner edge of the disc. If the star rotates with angular velocity Ω_*, this amount is reduced by a factor $1 - (\Omega_*/\Omega_K)^2$. Thus the star has to rotate close to break-up before the boundary-layer luminosity is much reduced. It is clear, therefore, that in the computation of the radiation spectrum of an accretion flow, via a disc, onto a star, the answer is incomplete unless some consideration is given to the spectrum of radiation from the boundary layer. Further discussion of stellar boundary layers is given in Lynden-Bell & Pringle (1974), Pringle (1977a), and in Pringle & Savonije (1979).

If the central star has a strong enough magnetic field, the disc flow can be disrupted at some distance from the star itself. In this case most of the accretion luminosity comes from the matter when it actually strikes the stellar surface (assuming that it is, in fact, accreted) and not from the disc. Recent discussions of the nature of the interaction between the disc and the magnetosphere are given by Ghosh & Lamb (1979), Anzer & Börner (1980), Arons & Lea (1980), and references therein.

If the accretion disc is around a black hole, the motion near the inner regions deviates from the simple Keplerian law. The equations are given by Novikov & Thorne (1973). It is generally assumed that the disc extends down to the innermost stable circular orbit R_{in} given by Bardeen et al. (1973). The assumption that the angular momenta of the disc and the hole are aligned is justified for the inner regions by the fact that the rotation of inertial frames close to the hole causes precession of the disc and eventual alignment (Bardeen & Petterson 1975). The further assumption that within the radius R_{in} the gas spirals into the hole without radiating is probably justified (Stoeger 1980). Thus, the energy released by the accreting matter corresponds to the binding energy at the radius R_{in}. For a Schwarzschild black hole ($a = 0$) this corresponds to about 6 percent of the rest mass energy and, for a maximal Kerr hole, to about 42 percent. For further details see the review by Eardley & Press (1975).

Recent attention on the inner regions of black hole accretion discs has concentrated on the case when the disc is relatively thick in the z-direction. In this case the inner edge of the disc can be cusp-shaped and there is a central vortex-shaped region along the $R = 0$ axis which is free of material (Abramowicz et al. 1978). The shape of the central vortex is completely determined by specifying the angular momentum distribution of the matter close to the center. Attention centers on the question of whether these nozzle-shaped spaces can give rise to twin jets of matter or radiation in opposite directions of the kind seen in objects as apparently dissimilar as the double radio sources and the stellar type objects Sco X-1 and SS433 (Lynden-Bell 1978). At the time of writing rapid progress is being made in this area.

10. SUMMARY

The major uncertainties in accretion disc theory are the dissipation process (viscosity) and the emission process. Disc modeling is in some sense more of an art than a science, and since a steady disc can be constructed for almost any combination of viscosity and radiation process the possibilities for extending one's list of publications are almost endless. Conversely we are likely to learn more about the properties of accretion discs in astrophysics by observation and subsequent modeling than by pure theorizing. Thus, although accretion discs may play a major role in quasars, galactic nuclei, binary X-ray sources, solar system formation, and the like, much useful information has yet to be derived from the accretion discs we can actually see, for example around some of the planets and in particular in binary star systems such as the cataclysmic variables.

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